

# Observable Quantities in Weyl Gravity

M.R. Tanhayi<sup>1\*</sup>, M. Fathi<sup>1†</sup>, M.V. Takook<sup>2‡</sup>

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<sup>1</sup> *Department of Physics, Islamic Azad University, Central Tehran Branch, Tehran, Iran*

<sup>2</sup> *Department of Physics, Razi university, Kermanshah, Iran*

## Abstract

In this paper, the cosmological “constant” and the Hubble parameter are considered in the Weyl theory of gravity, by taking them as functions of  $r$  and  $t$ , respectively. Based on this theory and in the linear approximation, we obtain the values of  $H_0$  and  $\Lambda_0$  which are in good agreement with the known values of the parameters for the current state of the universe.

## 1 Introduction

Einstein’s general theory of relativity seems to be a perfect theory (at least in the classical level) which almost all the classical tests confirm it. This theory is obtained from the Einstein-Hilbert action

$$I_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R,$$

where  $R$  is the Ricci scalar. However, in some astrophysical issues, we have to consider other sources of matter and energy with repulsive gravitational properties ([1] and Ref.s therein). For example in the rotation curves of spiral galaxies where the galactic rotational velocities differ from the velocities that predicted by the Newtonian gravitational potentials due to the luminous matter in the galaxies [2], another is related to the origin of the positive accelerating expansion of our universe. These unseen or dark sources are presently the most exciting open problems in cosmology and up to present days there is no satisfactory explanation for the origin of them. There are various theories to construct acceptable dark energy models [3], among these models, the modified gravity approach is attractive, since it tries to explain a natural gravitational alternative for dark sources. In this theory, the standard Einstein-Hilbert action is generalized to more complicated analytic function of the curvature ( $f(R)$ -gravity). There exist more fundamental reasons for consideration of such theories. For example the corrected gravitational potential which obtained can predict the Milky way rotation curve without the need of the dark matter scenario. Almost all these theories lead to the higher order derivative field equations.

In the Weyl theory of gravity the Einstein-Hilbert action is replaced by the square of the conformal Weyl tensor

$$I_W = -\alpha \int d^4x \sqrt{-g} C_{\mu\nu\rho\lambda} C^{\mu\nu\rho\lambda}, \quad (1)$$

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\*e-mail: m – tanhayi@iauctb.ac.ir

†e-mail: mohsen.fathi@gmail.com

‡e-mail: takook@razi.ac.ir

where  $C_{\mu\nu\rho\lambda}$  is the Weyl tensor

$$C_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} - \frac{1}{2}(g_{\mu\lambda}R_{\nu\rho} - g_{\mu\rho}R_{\nu\lambda} - g_{\nu\lambda}R_{\mu\rho} + g_{\nu\rho}R_{\mu\lambda}) + \frac{R}{6}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}).$$

Since there is no accepted system of sign convention in general relativity, different articles use different sign conventions. In this work we first review Weyl gravity with the usual sign convention of Ref. [4, 5], and then by taking the cosmological constant and the Hubble parameter as functions of  $r$  and  $t$  respectively, it is shown that in the Weyl gravity, the estimated values for these quantities are close to their measured values. In the appendix, some useful relations have been presented.

## 2 Weyl gravity

The action (1) can be written as follows

$$I_W = -\alpha \int d^4x \sqrt{-g} (R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3}R^2).$$

Since  $\sqrt{-g}(R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} - 4R^{\mu\nu} R_{\mu\nu} + R^2)$  is a total divergence (Gauss-Bonnet term), it does not contribute to the equation of motion and one can simplify the action as follows [6]

$$I_W = -2\alpha \int d^4x \sqrt{-g} \left( R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{3}R^2 \right) \equiv -2\alpha \int d^4x \left( \mathcal{W}_2 - \frac{1}{3}\mathcal{W}_1 \right). \quad (2)$$

The variation of the action with respect to  $g_{\alpha\beta}$  results in

$$\begin{aligned} \delta(\mathcal{W}_1) &= 2\sqrt{-g}R\delta R + R^2\delta\sqrt{-g} \\ &= \sqrt{-g}\nabla_\gamma A^\gamma + \sqrt{-g}\left(2\nabla^\alpha\nabla^\beta R - 2g^{\alpha\beta}\square R - 2RR^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}R^2\right)\delta g_{\alpha\beta}, \end{aligned} \quad (3)$$

where

$$A^\gamma \equiv 2g^{\alpha\beta}g^{\gamma\lambda}R(\nabla_\alpha\delta g_{\beta\lambda} - \nabla_\lambda\delta g_{\alpha\beta}) - 2g^{\gamma\beta}g^{\alpha\lambda}\nabla_\alpha R\delta g_{\beta\lambda} + 2g^{\alpha\beta}g^{\gamma\lambda}\nabla_\lambda R\delta g_{\alpha\beta},$$

and

$$\begin{aligned} \delta(\mathcal{W}_2) &= \sqrt{-g}\nabla_\gamma B^\gamma + \\ &\sqrt{-g}\left(\nabla_\rho\nabla^\alpha R^{\beta\rho} + \nabla_\rho\nabla^\beta R^{\alpha\rho} - \square R^{\alpha\beta} - g^{\alpha\beta}\nabla_\rho\nabla_\lambda R^{\rho\lambda} - 2R_\rho^\beta R^{\rho\alpha} + \frac{1}{2}g^{\alpha\beta}R_{\rho\lambda}R^{\rho\lambda}\right)\delta g_{\alpha\beta}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} B^\gamma &= g^{\gamma\rho}R^{\alpha\beta}\left(\nabla_\alpha\delta g_{\beta\rho} + \nabla_\beta\delta g_{\alpha\rho} - \nabla_\rho\delta g_{\alpha\beta}\right) - g^{\alpha\rho}\left(\nabla_\alpha R^{\gamma\beta}\delta g_{\beta\rho} - \nabla_\beta R^{\gamma\beta}\delta g_{\alpha\rho}\right) \\ &\quad - g^{\beta\rho}\left(R^{\alpha\gamma}\nabla_\alpha\delta g_{\beta\rho} + \nabla_\beta R^{\alpha\gamma}\delta g_{\alpha\rho}\right) + g^{\rho\gamma}\nabla_\rho R^{\alpha\beta}\delta g_{\alpha\beta}. \end{aligned}$$

When the equation of motion is considered, the first terms in the right hand side of Eq.s (3) and (4) have no contribution. Therefore we obtain

$$\begin{aligned} W_{\alpha\beta} &\equiv W_{\alpha\beta}^{(2)} - \frac{1}{3}W_{\alpha\beta}^{(1)} = \nabla^\rho\nabla_\alpha R_{\beta\rho} + \nabla^\rho\nabla_\beta R_{\alpha\rho} - \square R_{\alpha\beta} - g_{\alpha\beta}\nabla_\rho\nabla_\lambda R^{\rho\lambda} \\ &\quad - 2R_{\rho\beta}R_\alpha^\rho + \frac{1}{2}g_{\alpha\beta}R_{\rho\lambda}R^{\rho\lambda} - \frac{1}{3}\left(2\nabla_\alpha\nabla_\beta R - 2g_{\alpha\beta}\square R - 2RR_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}R^2\right). \end{aligned} \quad (5)$$

Therefore,  $W_{\alpha\beta} = 0$  are the vacuum Weyl field equations<sup>1</sup>.

In one of the most significant papers (Ref. [7]), the authors concern about conformal radial solutions to vacuum Weyl field equations ( $W^{rr} = 0$ ). In the so called article, a general metric has been derived, covering all spherically symmetric solutions to Einstein field equations. Moreover, this metric possesses an additional constant, which is claimed that can geometrically explain the effects of dark energy. This metric, as it is described in [7], also covers the de Sitter solution with the cosmological term. In this work we obtain similar results, by considering de Sitter like metric in the background field method and linear approximation.

### 3 An estimation for $\Lambda$ and $H$

Let us choose the following metric that mimics the de Sitter-Schwarzschild metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = - \left( 1 - 2 \frac{GM}{r} - \frac{1}{3} f(r) r^2 \right) dt^2 + \left( 1 - 2 \frac{GM}{r} - \frac{1}{3} f(r) r^2 \right)^{-1} dr^2 + r^2 d\Omega^2. \quad (6)$$

After doing some tedious but straightforward calculation, the components of the Ricci tensor become as follows:

$$\begin{aligned} R_{00} &= \frac{1}{2} \left( 1 - 2 \frac{GM}{r} - \frac{1}{3} f r^2 \right) \left( -\frac{1}{3} f'' r^2 - 2 f' r - 2 f \right), \\ R_{11} &= - \left( 1 - 2 \frac{GM}{r} - \frac{1}{3} f r^2 \right)^{-2} R_{00}, \\ R_{22} &= \frac{1}{3} f' r^3 + f r^2, \\ R_{33} &= R_{22} (\sin(\theta))^2. \end{aligned} \quad (7)$$

And the Ricci scalar takes the following form

$$R = \frac{1}{3} (f'' r^2 + 8 f' r + 12 f), \quad (8)$$

where the prime stands for the derivative with respect to  $r$ . And also the component of the Weyl equation is found as

$$\begin{aligned} W_{00} &= \frac{1}{9} (96 \frac{(GM)^2}{r^2} f'' r - 106 \frac{GM}{r} f'' + \frac{4}{3} f'^2 r GM + \frac{1}{18} f' r^6 f f''' + \frac{2}{9} f' r^5 f f'' + 42 \frac{(GM)^2}{r} f''' \\ &\quad - \frac{1}{6} r^3 f''^2 GM - 43 GM f''' - 4 f'''' r GM + 4 (GM)^2 f'''' - 44 \frac{GM}{r^2} f' r + 11 f''' r + 29 f'' \\ &\quad + 16 \frac{f'}{r} - \frac{2}{3} r^2 f'^2 + \frac{1}{12} r^4 f''^2 + f'''' r^2 - \frac{23}{3} f r^3 f''' + \frac{1}{9} f^2 r^6 f'''' - \frac{1}{6} r^4 f' f''' - \frac{40}{3} f r f' \\ &\quad + \frac{2}{9} f'^2 r^4 f + \frac{8}{3} f^2 r^3 f' - \frac{2}{3} f'''' r^4 f - \frac{1}{36} r^6 f f''^2 - \frac{2}{3} r^3 f' f'' + 24 \frac{(GM)^2}{r^3} f' + 4 f^2 r^4 f'' \\ &\quad - \frac{65}{3} f r^2 f'' + \frac{1}{3} f' r^3 GM f''' + 20 GM f f' + \frac{4}{3} f' r^2 GM f'' + 40 GM f r f'' \\ &\quad + 15 GM f r^2 f''' + \frac{4}{3} GM f'''' r^3 f + \frac{4}{3} f^2 r^5 f'''), \end{aligned} \quad (9)$$

$$W_{11} = \frac{1}{18(-3r + 6GM + f r^3)} \times$$

$$\left( -48 f' - 42 f'' r - r^5 f' f''' + 216 \frac{GM}{r} f' + 18 f''' r GM + 144 f'' GM - 6 f''' r^2 + \frac{1}{2} r^5 f''^2 - 4 f' r^4 f'' - 4 f'^2 r^3 \right), \quad (10)$$

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<sup>1</sup>The relation (5) has been written by taking the following sign convention of [4]

$$R_{\beta\gamma\rho}^\alpha = \partial_\gamma \Gamma_{\beta\rho}^\alpha + \Gamma_{\lambda\gamma}^\alpha \Gamma_{\beta\rho}^\lambda - (\gamma \leftrightarrow \rho),$$

$$R_{\beta\rho} = R_{\beta\alpha\rho}^\alpha, \text{ and } R = R_\alpha^\alpha,$$

in Ref. [6], the opposite sign for the Ricci tensor and Ricci scalar has been chosen.

$$\begin{aligned}
9W_{22} &= \frac{1}{12}r^6f''^2 - \frac{2}{3}f'^2r^4 + \frac{1}{2}f''''r^4 + 5f'''r^3 + 11f''r^2 + 4f'r - 4f r^3f' + 12GMf' \\
&- \frac{1}{6}r^6f'f'''' - \frac{2}{3}f'r^5f'' - 6f''r^4f - \frac{1}{6}f''''r^6f - 2f'''r^5f - 12f''rGM - f''''r^3GM \\
-9f'''r^2GM &= \frac{9}{\sin^2(\theta)}W_{33}.
\end{aligned} \tag{11}$$

Letting  $W_{\alpha\beta} = 0$  results in four differential equations but here, the common solution for  $f(r)$  is considered (the general form of the solution can be found in [7])

$$f(r) = c_1 + \frac{6GM}{r^3}. \tag{12}$$

Taking into account this value for  $f(r)$  in (6), yields

$$g_{00} = -\left(1 - \frac{4GM}{r} - \frac{1}{3}c_1r^2\right). \tag{13}$$

This is similar to the de Sitter-Schwarzschild metric (except the coefficient of  $1/r$ ) that indicate this metric is a solution as well.

Now let us use the weak field limit which implies

$$g_{00} = \eta_{00} + h_{00}, \tag{14}$$

where  $\eta_{00} = -1$ , and  $h_{00} = \frac{4GM}{r} + \frac{1}{3}c_1r^2$  is considered to be very small. The Poisson equation for this potential implies that

$$\nabla^2 h_{00} = 8\pi T_{00}, \tag{15}$$

in which  $T_{00}$  is the density of a homogenous spherical observed mass,  $M_{obs}$ , with the observed radius,  $r_{obs}$ . From (13) and (15), it follows that

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\left(\frac{4GM}{r} + \frac{1}{3}c_1r^2\right) = 8\pi\rho_{obs}, \tag{16}$$

where  $\rho_{obs} = \frac{3M_{obs}}{4\pi r_{obs}^3}$ . So we find

$$c_1 = 3\frac{M_{obs}}{r_{obs}^3}. \tag{17}$$

This relation may be considered as an estimation for the cosmological constant (see relation (13) with  $c_1 \equiv \Lambda$ ). Taking  $r_{ob} \approx 4.39 \times 10^{28} \text{ cm}$  [8, 9] and  $M_{ob} \approx 8 \times 10^{55} \text{ gr}$ , in which  $r_{obs}$  is the radius of the observable universe and  $M_{obs}$  is the mass amount of a steady-state observable universe which has been calculated by Sir Fred Hoyle [10], we obtain

$$c_1 \equiv \Lambda = 3\frac{M_{ob}}{r_{ob}^3} \approx 2.84 \times 10^{-30} \frac{\text{gr}}{\text{cm}^3}.$$

This can be comparable to the value, which has been estimated for the cosmological constant  $\Lambda_0$ . Note that the cosmological constant is measured to be of order  $10^{-29} \frac{\text{gr}}{\text{cm}^3}$  [11].

At the end of this section, let us assume a general time dependent metric as:

$$ds^2 = -dt^2 + f(t)dx_i^2, \tag{18}$$

in which  $f(t)$  is an arbitrary function of  $t$ . It is found that this general time-dependent metric, is a solution of Eq. (5) ( $W_{\alpha\beta} = 0$ ). Therefore we rewrite the metric (18) as follows

$$ds^2 = -dt^2 + e^{2h(t)t}dx_i^2. \tag{19}$$

Using this metric, the Ricci tensor components become as follows:

$$\begin{aligned} R_{00} &= -3\left(\ddot{h}t + 2\dot{h}\right) + (\dot{h}t + h)^2, \\ R_{11} &= R_{22} = R_{33} = e^{2ht}\left(\ddot{h}t + 2\dot{h}\right) + 3(\dot{h}t + h)^2. \end{aligned} \quad (20)$$

Also the Ricci scalar is calculated as follows

$$R = 6(\ddot{h}t + 2\dot{h} + 2\dot{h}^2t^2 + 4\dot{h}th + 2h^2), \quad (21)$$

where over dot presents the time derivative. Note that if  $h(t)$  is supposed to be a constant then the de Sitter condition will be obtained [4, 5].

Using these values in the expressions for  $W_{\alpha\beta}^{(1)}$  and  $W_{\alpha\beta}^{(2)}$  in Eq. (5) gives:

$$\begin{aligned} W_{00}^{(1)} &= -36\dot{h}t\ddot{h} + 72\dot{h}^2 + 18\ddot{h}^2t^2 - 36\dot{h}t^2h^{(3)} - 36h^{(3)}th - 108\ddot{h}h - 216\ddot{h}t^2\dot{h}h \\ &\quad - 216\dot{h}^3t^2 - 216\dot{h}h^2 - 108\ddot{h}t^3\dot{h}^2 - 108\ddot{h}th^2 - 432\dot{h}^2th, \\ W_{00}^{(2)} &= 24\dot{h}^2 - 72\ddot{h}t^2\dot{h}h - 12\dot{h}t\ddot{h} - 12\dot{h}t^2h^{(3)} - 12h^{(3)}th - 36\ddot{h}t^3\dot{h}^2 - 36\ddot{h}th^2 - 144\dot{h}^2th \\ &\quad + 6\ddot{h}^2t^2 - 36\ddot{h}h - 72\dot{h}^3t^2 - 72\dot{h}h^2. \end{aligned} \quad (22)$$

From the above relations it is clear that  $W_{00} (= W_{00}^{(2)} - \frac{1}{3}W_{00}^{(1)})$  becomes zero for all  $h(t)$ . Thus let us consider the following  $h(t)$

$$h(t) = h_1 \frac{\ln t}{t} + h_2,$$

where  $h_1$  and  $h_2$  are two constants. If we take  $a^2(t) = e^{2h(t)t}$ , where  $a(t)$  stands for the scale factor, then the Hubble parameter reads as follows

$$H = \frac{\dot{a}}{a} = \frac{h_1}{t} + h_2.$$

As a simple guess in this work, we take  $h_1 = 1$  and  $h_2 = 0$ . And if  $t$  be assumed the age of the universe - the age of the universe is now calculated to be  $13.75 \times 10^9 \pm 0.17$  yrs [12] - then one obtains the Hubble constant for this time as:

$$H \approx 2.31 \times 10^{-18} s^{-1}. \quad (23)$$

The recent measurement for Hubble constant is  $H_0 = 73.8 \pm 2.4$  (km/s)/Mpc or  $2.39 \pm 8\% \times 10^{-18} s^{-1}$  [13].

## 4 Conclusion

Conformal symmetry is indeed one of the most important measures of assessment of massless field in quantum field theory. If the graviton does exist and propagates on the light cone due to its long range effect, it should have zero mass. The light cone propagation immediately imposes the conformal invariance on the graviton field equations. Historically, the first conformally invariant theory of gravity introduced by Weyl in 1918; then Einstein pointed out that the non-integrability of the lengths of vectors under Weyl-like parallel propagation contradicts to physical experience (very good bibliography and background are given in [14]). In recent years there is a renewal interest of such scale-invariant theories. For example Friedmann equation of cosmological evolution for this theory can be found in [15, 16]. In Weyl gravity, the usual Einstein-Hilbert action, is replaced by the square of the conformal Weyl tensor. This leads to a gravitational theory of fourth order. Previously, exact vacuum solution to conformal Weyl gravity had been

studied in [7], however in this paper, it has been shown that some significant radial solutions, (such as de Sitter and Reissner-Nordström metrics), in the linear approximation and the background field method, can be easily obtained. We then studied the time and  $r$  dependence of the Hubble and cosmological parameters in the Weyl theory of gravity. As a result, one may obtain theoretically some observable quantities, with high accuracy, in higher order theories of gravity (including the Weyl theory), as it has been confirmed by other authors [17, 18].

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## A appendix

Recent astrophysical data indicate that our universe has a positive and non-vanishing acceleration, therefore it might well be explained by de Sitter model. De Sitter space-time can be considered as a vacuum solution of the Weyl gravity. Considering the condition

$$R_{\mu\nu} = \frac{1}{4}g_{\mu\nu}R, \quad (24)$$

together with the Bianchi identity, one obtains  $\nabla_\mu R = 0$ , it means that  $R$  is a constant. Imposing this condition into the Einstein's equation, leads to

$$R = 4\Lambda. \quad (25)$$

This is the well-known de Sitter condition, which is followed by a constant  $R$ . De Sitter space-time is maximally symmetric and we have [4, 5],

$$R_{abcd} = H^2(g_{ac}g_{bd} - g_{ad}g_{bc}), \quad R_{ab} = 3H^2g_{ab}, \quad R = 12H^2 = 4\Lambda.$$

The following relations become important in writing the Weyl gravity in de Sitter background and in the linear form

$$\begin{aligned} \delta\Gamma_{bd}^a &= \frac{1}{2}g^{ae}\left(\nabla_b\delta g_{ed} + \nabla_d\delta g_{be} - \nabla_e\delta g_{bd}\right), \\ \delta R_{bcd}^a &= \frac{1}{2}g^{ae}\left(\nabla_c\nabla_b\delta g_{de} + \nabla_c\nabla_d\delta g_{be} - \nabla_c\nabla_e\delta g_{bd} - \nabla_d\nabla_b\delta g_{ce} - \nabla_d\nabla_c\delta g_{be} + \nabla_d\nabla_e\delta g_{bc}\right), \\ \delta R_{ab} &= \frac{1}{2}g^{cd}\left(\nabla_c\nabla_a\delta g_{bd} + \nabla_c\nabla_b\delta g_{ad} - \nabla_c\nabla_d\delta g_{ab} - \nabla_b\nabla_a\delta g_{cd}\right), \\ \delta R &= \left(\nabla^a\nabla^b\delta g_{ab} - g^{ab}\square\delta g_{ab}\right) - 3H^2g^{ab}\delta g_{ab}. \end{aligned}$$

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